We want to find $\langle m(\vec{x})\, m(\vec{x}')\rangle$, and for simplicity we focus at $T > T_c$ ($\langle m(\vec{x})\rangle = 0$). Otherwise we should write $m(\vec{x}) = m^* + \mu(\vec{x})$ and the free energy reduces to a function(al) of $\mu(\vec{x})$

Let's go back to the partition function

$$Z = \int \mathcal{D}m(\vec{x})\, e^{N\int d\vec{x}\left[\frac{1}{2} t\, m^2 + \frac{1}{2} b(\vec{\nabla} m)^2\right]}$$

where

$$\left.\begin{aligned} t &= \frac{T-T_c}{T_c} \\ b &= \frac{J' z}{k_B T} \end{aligned}\right\} \text{close to } T_c$$

Let's go to Fourier space : $m(\vec{x}) \longleftrightarrow \tilde{m}(\vec{k})$

$m(\vec{x})$ and $\tilde{m}(\vec{k})$ are two perfectly equivalent representations of the field m.

Let's use the convention

$$\tilde{m}(\vec{k}) = \int d\vec{x}\; e^{-i\vec{k}\cdot\vec{x}}\, m(\vec{x})$$

$$m(\vec{x}) = \frac{1}{(2\pi)^d} \int d\vec{k}\; e^{i\vec{k}\vec{x}}\, \tilde{m}(\vec{k})$$

Then we have :

$m(\vec{x})$ is real : $m(\vec{x}) = \overset{\text{complex conjugate}}{m^*}(\vec{x})$

$$\Rightarrow \quad m(\vec{x}) = \frac{1}{(2\pi)^d}\int d\vec{k}\, e^{i\vec{k}\vec{x}}\,\tilde{m}(\vec{k}) \overset{\text{conjugate}}{=} \frac{1}{(2\pi)^d}\int d\vec{k}\, e^{-i\vec{k}\vec{x}}\,\tilde{m}^*(\vec{k}) =$$

$$\underset{\vec{k}\to-\vec{k}}{=} \frac{1}{(2\pi)^d}\int d\vec{k}\, e^{i\vec{k}\vec{x}}\,\tilde{m}^*(-\vec{k})$$

From which we have $\tilde{m}(\vec{k}) = \tilde{m}^*(-\vec{k})$

Moreover :

$$\vec{\nabla} m = \vec{\nabla}\frac{1}{(2\pi)^d}\int d\vec{k}\, e^{i\vec{k}\vec{x}}\,\tilde{m}(\vec{k}) = \frac{i}{(2\pi)^d}\int d\vec{k}\,\vec{k}\, e^{i\vec{k}\vec{x}}\,\tilde{m}(\vec{k})$$

$$\Rightarrow \quad \mathcal{F}[\vec{\nabla} m] = i\vec{k}\,\tilde{m}(\vec{k})$$

Now we write

$$m^2(\vec{x}) = \left[\frac{1}{(2\pi)^d}\right]^2 \int d\vec{k}\int d\vec{k}'\, e^{i\vec{k}\vec{x}}\, e^{i\vec{k}'\vec{x}}\,\tilde{m}(\vec{k})\,\tilde{m}(\vec{k}')$$

and

$$\int d\vec{x}\; m^2(\vec{x}) = \left[\frac{1}{(2\pi)^d}\right]^2 \int d\vec{k}\int d\vec{k}'\,\tilde{m}(\vec{k})\,\tilde{m}(\vec{k}') \cdot \int d\vec{x}\; e^{i(\vec{k}+\vec{k}')\vec{x}} =$$

$$= \frac{1}{(2\pi)^d}\int d\vec{k}\, d\vec{k}'\,\tilde{m}(\vec{k})\,\tilde{m}(\vec{k}')\;\delta(\vec{k}+\vec{k}') =$$

$$= \frac{1}{(2\pi)^d}\int d\vec{k}\;\tilde{m}(\vec{k})\,\tilde{m}(-\vec{k}) = \frac{1}{(2\pi)^d}\int d\vec{k}\;\tilde{m}(\vec{k})\,\tilde{m}^*(\vec{k})$$

We treat in the same way $(\nabla \vec{m})^2$:

$$(\nabla \vec{m})^2 = \left[\frac{1}{(2\pi)^d}\right]^2 \int d\vec{k} \int d\vec{k}' \, e^{i(\vec{k}+\vec{k}')\vec{x}} (i\vec{k})(i\vec{k}') \tilde{m}(\vec{k})\, \tilde{m}(\vec{k}')$$

then

$$\int d\vec{x}\, (\nabla \vec{m})^2 = \frac{1}{(2\pi)^d} \int d\vec{k}\; \vec{k}^2\, \tilde{m}(\vec{k})\, \tilde{m}^*(\vec{k})$$

The Partition function is thus

$$Z = \int \mathcal{D}\tilde{m} \; e^{-\frac{NV}{(2\pi)^d}\int d\vec{k} \left[\frac{1}{2}(t + bk^2)\, \tilde{m}(\vec{k})\, \tilde{m}^*(\vec{k})\right]}$$

Now let's write $m(\vec{x})\, m(\vec{x}')$ using Fourier

$$m(\vec{x})\, m(\vec{x}') = m(\vec{x})\, m(\overbrace{\vec{x} + \vec{r}}^{\vec{x}' = \vec{x} + \vec{r}}) =$$

$$= \frac{1}{(2\pi)^{2d}} \int d\vec{k} \int d\vec{k}' \; e^{i\vec{k}\vec{x} + i\vec{k}'(\vec{x} + \vec{r})}\, \tilde{m}(\vec{k})\, \tilde{m}(\vec{k}') =$$

$$= \frac{1}{(2\pi)^{2d}} \int d\vec{k} \int d\vec{k}' \; e^{i(\vec{k} + \vec{k}')\vec{x} + i\vec{k}'\vec{r}}\, \tilde{m}(\vec{k})\, \tilde{m}(\vec{k}')$$

Actually, we can compute the correlation averaged over space:

$$c(\vec{r}) = \frac{1}{V}\int d\vec{x}\; m(\vec{x})\,\vec{m}(\vec{x}+\vec{r}) =$$

$$= \frac{1}{V}\frac{1}{(2\pi)^{2d}}\int d\vec{k}\int d\vec{k}'\; \tilde{m}(\vec{k})\,\tilde{m}(\vec{k}')\,e^{i\vec{k}'\vec{r}}\underbrace{\int e^{i(\vec{k}+\vec{k}')\vec{x}}d\vec{x}}_{2\pi\,\delta(\vec{k}+\vec{k}')}$$

$$= \frac{1}{V}\frac{1}{(2\pi)}\int d\vec{k}\int d\vec{k}'\,\tilde{m}(\vec{k})\,\tilde{m}(\vec{k}')\,e^{i\vec{k}'\vec{r}}\,\delta(\vec{k}+\vec{k}') =$$

$$= \frac{1}{V}\frac{1}{2\pi}\int d\vec{k}\;\tilde{m}(\vec{k})\,\tilde{m}(-\vec{k})\,e^{-i\vec{k}\cdot\vec{r}} =$$

$$= \frac{1}{V}\frac{1}{2\pi}\int d\vec{k}\;\tilde{m}(\vec{k})\,\tilde{m}^*(\vec{k})\,e^{-i\vec{k}\cdot\vec{r}}$$

At last we can put everything together:

$$\langle c(\vec{r})\rangle = \frac{1}{V}\frac{1}{2\pi}\int \underset{\uparrow}{\mathcal{D}\tilde{m}}\int d\vec{k}\;\tilde{m}(\vec{k})\,\tilde{m}^*(\vec{k})\,e^{-i\vec{k}\cdot\vec{r}}\;\frac{e^{-\frac{NV}{(2\pi)^d}\int d\vec{k}\left[\frac{1}{2}(t+bk^2)\,\tilde{m}\,\tilde{m}^*\right]}}{Z}$$

integrating over all possible $m(\vec{x})$ or all possible $\tilde{m}(\vec{k})$ is the same thing because they are in correspondence due to Fourier (it is just a change of variable, if you want)

$$\langle c(\vec{r})\rangle = \frac{1}{V}\frac{1}{2\pi}\int \mathcal{D}\tilde{m}_R\, \mathcal{D}\tilde{m}_I \int (\tilde{m}_R^2+\tilde{m}_I^2)\cdot \frac{e^{-NV\int \frac{1}{2}[t+bk^2](\tilde{m}_R^2+\tilde{m}_I^2)\,d\vec{k}}}{Z}\cdot$$

$$\cdot\, e^{-i\vec{k}\cdot\vec{r}}\, d\vec{k}$$

A few considerations:

1) $$\mathcal{D}m \longrightarrow \mathcal{D}\tilde{m}_R\, \mathcal{D}\tilde{m}_I$$ seems like doubling the variables!

It is not, because $\tilde{m}_R(\vec{k}) = \tilde{m}_R(-\vec{k})$ and $\tilde{m}_I(\vec{k}) = -\,\tilde{m}_I(-\vec{k})$

so that it is divided by two in $\vec{k}$ space

2) The integrals for $\tilde{m}_R$ and $\tilde{m}_I$ will be identical

3) How do we perform that integral? Let's learn how to manipulate it using a bit of intuition

$\int \mathcal{D}\tilde{m}_R\, \mathcal{D}\tilde{m}_I$ runs over all possible values that each $\tilde{m}_R$ and $\tilde{m}_I$ can take for each $\vec{k}$

Let's write it as a sum, if $\vec{k}$ was discrete (for finite volume, actually, it is).

Then

$$\int \mathcal{D}\tilde{m}_R \, \mathcal{D}\tilde{m}_I \longrightarrow \int_{-\infty}^{+\infty} \prod_{\vec{k}} d\tilde{m}_R(\vec{k}) \int_{-\infty}^{+\infty} \prod_{\vec{k}} d\tilde{m}_I(\vec{k})$$

Likewise the free-energy (argument of the exponential) is:

$$\int d\vec{k} \left[\frac{1}{2}(t+b^2k^2)(\tilde{m}_R^2 + \tilde{m}_I^2) \right] \longrightarrow \sum_{\vec{k}} \left[\frac{1}{2}(t+bk^2)(\tilde{m}_R^2 + \tilde{m}_I^2) \right]$$

The partition function is thus

$$Z = \int \mathcal{D}\tilde{m}_R \, \mathcal{D}\tilde{m}_I \; e^{-NV\int d\vec{k} \, \frac{1}{2}(t+bk^2)(\tilde{m}_R^2 + \tilde{m}_I^2)}$$

$$\downarrow$$

$$\int_{-\infty}^{+\infty} \prod_{\vec{k}} d\tilde{m}_R(\vec{k}) \int_{-\infty}^{+\infty} \prod_{\vec{k}} d\tilde{m}_I(\vec{k}) \; \prod_{\vec{k}} e^{-\frac{1}{2}(t+bk^2)(\tilde{m}_R^2 + \tilde{m}_I^2)} =$$

$$= \prod_{\vec{k}} I^2(\vec{k}) \qquad \text{with} \qquad I(\vec{k}) = \int_{-\infty}^{+\infty} dm \; e^{-\frac{1}{2}(t+bk^2)m^2}$$

We can now compute $\langle c(\vec{r}) \rangle$:

$$\langle c(\vec{r}) \rangle = \frac{1}{(2\pi)^d} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \prod_{\vec{k}'} d\tilde{m}_R(\vec{k}') \int_{-\infty}^{+\infty} \prod_{\vec{k}'} d\tilde{m}_I(\vec{k}') \; \tilde{m}_R^2(\vec{k}) \frac{\prod_{\vec{k}'} e^{-NV\left[\frac{1}{2}(t+bk^2)(\tilde{m}_R^2(\vec{k}') + \tilde{m}_I^2(\vec{k}'))\right]}}{\prod_{\vec{k}'} I^2(\vec{k}')} \cdot$$

$$\cdot \; e^{-i\vec{k}\cdot\vec{r}} \, d\vec{k} \; +$$

$$+ \text{ same for } \tilde{m}_I^2(\vec{k})$$

Thus

$$\langle c(\vec{r})\rangle = \left(\frac{1}{2\pi}\right)^d \int_{-\infty}^{+\infty} d\vec{k}\, e^{-i\vec{k}\cdot\vec{r}} \int_{-\infty}^{+\infty} d\tilde{m}_R(\vec{k})\, \frac{\prod_{\vec{k}'\neq\vec{k}} I^2(\vec{k}')\, I(\vec{k}) \cdot \tilde{m}_R^2(\vec{k})\, e^{-NV\left[\frac{1}{2}(t+bk^2)\tilde{m}_R^2\right]}}{\prod_{\vec{k}'} I^2(\vec{k}')} =$$

$+$ same for $\tilde{m}_I$

The final result for $\tilde{m}_R$ and $\tilde{m}_I$ is the same

$$= \frac{2}{(2\pi)^d} \int_{-\infty}^{+\infty} d\vec{k}\, e^{-i\vec{k}\cdot\vec{r}} \int_{-\infty}^{+\infty} d\tilde{m}_R(\vec{k})\ \tilde{m}_R^2(\vec{k})\ \underbrace{\frac{e^{-NV\left[\frac{1}{2}(t+bk^2)\tilde{m}_R^2\right]}}{I(\vec{k})}}_{\text{this is a properly normalized Gaussian}} =$$

$$= \frac{2}{(2\pi)^d} \int_{-\infty}^{+\infty} d\vec{k}\, e^{-i\vec{k}\cdot\vec{r}}\, \frac{1}{NV(t+bk^2)} =$$

$$= \frac{2}{NV} \frac{1}{(2\pi)^d} \int_{-\infty}^{+\infty} d\vec{k}\, e^{-i\vec{k}\cdot\vec{r}}\, \frac{1}{t+bk^2} =$$

$$= \frac{2}{NVb} \frac{1}{(2\pi)^d} \int_{-\infty}^{+\infty} d\vec{k}\, e^{-i\vec{k}\cdot\vec{r}}\, \frac{1}{\frac{t}{b}+k^2}$$

$$\langle c(\vec{r})\rangle = \frac{2}{NVb} \frac{1}{(2\pi)^d} \int_{-\infty}^{+\infty} d\vec{k}\, e^{-i\vec{k}\cdot\vec{r}}\, \frac{1}{\frac{t}{b}+k^2}$$

We thus have that the Fourier transform of $\langle c(\vec{r})\rangle$ is

$$\mathcal{F}\left[\langle c(\vec{r})\rangle\right] \propto \frac{1}{\frac{t}{b} + k^2} = \tilde{f}(\vec{k})$$

which is real and even, that is, $\tilde{f}(\vec{k}) = \tilde{f}^*(-\vec{k})$, and, as a consequence, $\langle c(\vec{r})\rangle$ is real (as expected)

The antitransform is (apart from multiplicative constants), in 3d (see exercise session)

$$\langle c(\vec{r})\rangle \propto \frac{1}{4\pi r} e^{-r/\xi}$$

where ξ is the correlation length $\xi = \sqrt{\frac{b}{t}} = \sqrt{b}\, t^{-1/2}$

In general it can be shown that

$$\langle c(\vec{r})\rangle \propto \frac{1}{r^{d-2}} F\left(\frac{r}{\xi}\right)$$

always with $\xi \propto t^{-1/2}$
and $F(x) \underset{x\to\infty}{\sim} e^{-x}$

(Mathematically much heavier to derive in $d \neq 3$; careful in

$d=2 \Rightarrow \frac{1}{r^{d-2}} \rightarrow \ln r$!

$$r^{2-d} \underset{d=2+\varepsilon}{\simeq} r^{-\varepsilon} \simeq e^{-\varepsilon \ln r} = 1 - \varepsilon \ln r$$)

It is remarkable that $\xi \sim t^{-1/2} \longrightarrow \infty$ if $t \to 0$ $(T \to T_c)$!!!

We have now the list of mean-field critical exponents:

	$m \sim t^{1/2}$	$\beta = 1/2$	$m \sim t^{\beta}$
	$m \sim h^{1/3}$	$\delta = 3$	$m \sim h^{1/\delta}$
specific heat	$C \sim t^{-\alpha}$	$\alpha = 0$	we did not derive it explicitly
	$\chi \sim t^{-1}$	$\gamma = 1$	$\chi \sim t^{-\gamma}$
	$\xi \sim t^{-1/2}$	$\nu = 1/2$	$\xi \sim t^{-\nu}$

Then there is

$$\langle c(\vec{r}) \rangle \sim \frac{1}{r^{d-2+\eta}} e^{-r/\xi} \quad \Rightarrow \quad \eta = 0$$